MATH 4030 Differential Geometry
Tutorial 8, 8 November 2017

1. Recall from the Lecture notes (Part 6 p.12) the *Gauss* and *Codazzi* equations:

$$\begin{cases} \partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^p_{ij}\Gamma^\ell_{pk} - \Gamma^p_{ik}\Gamma^\ell_{pj} = & g^{\ell p}(A_{ij}A_{kp} - A_{ik}A_{jp}) \\ \partial_k A_{ij} - \partial_j A_{ik} + \Gamma^p_{ij}A_{pk} - \Gamma^p_{ik}A_{pj} = & 0 \end{cases}.$$

Assume $X : (-1,1)^2 \to \mathbb{R}^3$ is a chart for a surface S with first fundamental form given by

$$(g_{ij}) = \begin{pmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{pmatrix}.$$

(a) Compute all the Christoffel symbols Γ^k_{ij}.

(b) Find the Gauss curvature K.

(c) Is it possible that the second fundamental form is given by

$$(A_{ij}) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}?$$

Solution.

(a) First to smooth our calculations, we replace the dummy variables (u, v) by (x_1, x_2). Recall that the Christoffel symbols Γ^k_{ij} are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

We need to find (g^{ij}) which is the inverse of (g_{ij}). It is given by $g^{ij} = \dfrac{1}{(1+x_1^2+x_2^2)^2}\delta_{ij}$. Note also that $g_{ij} = (1+x_1^2+x_2^2)^2\delta_{ij}$. Plugging everything needed into the above formula, we have

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2} \cdot \frac{1}{(1+x_1^2+x_2^2)^2}\delta_{k\ell}\left[4x_i(1+x_1^2+x_2^2)\delta_{\ell j} + 4x_j(1+x_1^2+x_2^2)\delta_{i\ell} - 4x_\ell(1+x_1^2+x_2^2)\delta_{ij}\right] \\ &= \frac{2}{1+x_1^2+x_2^2}[x_i\delta_{kj} + x_j\delta_{ik} - x_k\delta_{ij}], \end{aligned}$$

so that

$$\begin{cases} \Gamma^1_{11} = \dfrac{2x_1}{1+x_1^2+x_2^2} \\ \Gamma^1_{12} = \dfrac{2x_2}{1+x_1^2+x_2^2} \\ \Gamma^1_{22} = \dfrac{-2x_1}{1+x_1^2+x_2^2} \end{cases} \quad \text{and} \quad \begin{cases} \Gamma^2_{11} = \dfrac{-2x_2}{1+x_1^2+x_2^2} \\ \Gamma^2_{12} = \dfrac{2x_1}{1+x_1^2+x_2^2} \\ \Gamma^2_{22} = \dfrac{2x_2}{1+x_1^2+x_2^2} \end{cases}.$$

(b) Note that the Gauss equation is equivalent to

$$g_{q\ell}(\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^p_{ij}\Gamma^\ell_{pk} - \Gamma^p_{ik}\Gamma^\ell_{pj}) = A_{ij}A_{kq} - A_{ik}A_{jq}.$$

This can be seen by multiplying $g_{q\ell}$ both sides, and summing over ℓ.

Now put $i = j = 1,\ k = q = 2$. we have RHS $= K \cdot \det(g_{ij}) = K(1 + x_1^2 + x_2^2)^4$ and LHS

$$
\begin{aligned}
&= g_{2\ell}(\partial_2 \Gamma^\ell_{11} - \partial_1 \Gamma^\ell_{12} + \Gamma^p_{11}\Gamma^\ell_{p2} - \Gamma^p_{12}\Gamma^\ell_{p1}) \\
&= 2(1 + x_1^2 + x_2^2)^2 \left[\frac{-1 - x_1^2 - x_2^2 + 2x_2^2}{(1 + x_1^2 + x_2^2)^2} - \frac{1 + x_1^2 + x_2^2 - 2x_1^2}{(1 + x_1^2 + x_2^2)^2} + \frac{2x_1^2 - 2x_2^2}{(1 + x_1^2 + x_2^2)^2} - \frac{-2x_2^2 + 2x_1^2}{(1 + x_1^2 + x_2^2)^2} \right] \\
&= -4.
\end{aligned}
$$

It follows that $K = \dfrac{-4}{(1 + x_1^2 + x_2^2)^4}$.

(c) We may try to show this is impossible, so we check whether the given (A_{ij}) is compatible with the Gauss and Codazzi equations or not, but

(G) $\det \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = -4$ which is equal to LHS in the last part. (We also put $i = j = 1,\ k = q = 2$.) That means the given data do not contradict the Gauss equation.

(C) $\partial_2 A_{11} - \partial_1 A_{12} + \Gamma^p_{11} A_{p2} - \Gamma^p_{12} A_{p1} = \left(\dfrac{-2x_2}{1 + x_1^2 + x_2^2} \right)(2) - \left(\dfrac{2x_2}{1 + x_1^2 + x_2^2} \right)(-2) = 0$ (where $i = j = 1,\ k = 2$). We can see that the Codazzi equation is also satisfied for other choices of i, j, k. That means the given data do not contradict the Codazzi equation neither.

In other words, we cannot find anything wrong by checking the compatibility of the Gauss and Codazzi equations. We may then turn to attempt to show that it is affirmative. Notice that the shape operator is

$$
\mathcal{S} = g^{-1}A = \frac{2}{(1 + x_1^2 + x_2^2)^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

which has zero trace. In other words, S is minimal. So we look at the list of some well-known minimal surfaces on the Internet and find out that the following surface satisfies the condition (we change (x_1, x_2) back to (u, v)):

$$
X : (-1, 1)^2 \to \mathbb{R}^3 : (u, v) \mapsto \left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - u^2 v, u^2 - v^2 \right).
$$

To see this, we compute

$$
\begin{aligned}
X_u &= (1 - u^2 + v^2, -2uv, 2u) \\
X_v &= (2uv, -1 + v^2 - u^2, -2v) \\
N &= \frac{1}{1 + u^2 + v^2}(2u, 2v, u^2 + v^2 - 1) \\
X_{uu} &= (-2u, -2v, 2) \\
X_{uv} &= (2v, -2u, 0) \\
X_{vv} &= (2u, 2v, -2)
\end{aligned}
$$

$$
\begin{aligned}
\langle X_{uu}, N \rangle &= \frac{1}{1 + u^2 + v^2}(-4u^2 - 4v^2 + 2u^2 + 2v^2 - 2) = -2 \\
\langle X_{uv}, N \rangle &= \frac{1}{1 + u^2 + v^2}(4uv - 4uv) = 0 \\
\langle X_{vv}, N \rangle &= \frac{1}{1 + u^2 + v^2}(4u^2 + 4v^2 - 2u^2 - 2v^2 + 2) = 2
\end{aligned}
$$

so that (A_{ij}) is indeed the given matrix. This surface is called the *Enneper surface*.